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Bifurcation surfaces stemming from the Fučik spectrum

Chong Li [a,*,1], Shujie Li [a,b,2], Zhaoli Liu [c,3]

[a] *Institute of Mathematics, AMSS, Academia Sinica, Beijing 100190, China*
[b] *Y.-R. Zeng Research Center, Harbin Normal University, Harbin 150025, China*
[c] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

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Abstract

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ and $g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a nonlinear function. We prove existence of two-dimensional bifurcation surfaces for the elliptic boundary value problem

$$-\Delta u = au^- + bu^+ + g(x, u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

where $u^- = \min\{0, u\}$, $u^+ = \max\{0, u\}$, and $(a, b) \in \mathbb{R}^2$ is a pair of parameters. We show that these two-dimensional bifurcation surfaces stem from the Fučik spectrum of $-\Delta$. The main difficulty in doing that comes from non-smoothness of the operators $u \mapsto u^\pm$. In order to overcome this difficulty, a variant implicit function theorem and an abstract two-dimensional bifurcation theorem are proved. These two theorems do not require smoothness of operators and the abstract two-dimensional bifurcation theorem can be regarded as an extension of the well-known Crandall–Rabinowitz bifurcation theorem, and therefore are of interest for their own sake.

Keywords: Implicit function theorem; Bifurcation theorem; Fučik spectrum; Bifurcation surface

* Corresponding author.
E-mail address: lichong@amss.ac.cn (C. Li).

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1. Introduction

In 1971, M.G. Crandall and P.H. Rabinowitz proved the following bifurcation theorem in [5] (see also [3,6,23]).

Theorem A *(Crandall–Rabinowitz). Let X, Y be Banach spaces and $U \subset X$ an open neighborhood of* 0. *Assume $F \in C^2(U \times \mathbb{R}, Y)$ satisfies $F(0,\lambda) = 0$, $F_x(0,\lambda_0)$ is a Fredholm operator with* $\dim\ker F_x(0,\lambda_0) = \operatorname{codim}\operatorname{Im} F_x(0,\lambda_0) = 1$, *and $F_{x\lambda}(0,\lambda_0)u_0 \notin \operatorname{Im} F_x(0,\lambda_0)$ where u_0 is such that* $\operatorname{span}\{u_0\} = \ker F_x(0,\lambda_0)$. *Let Z be any complement of* $\operatorname{span}\{u_0\}$ *in X. Then there exist $\delta > 0$ and a unique C^1 curve $(\lambda,\psi) : (-\delta,\delta) \to \mathbb{R} \times Z$ satisfying*

$$\begin{cases} F\big(su_0 + s\psi(s), \lambda(s)\big) = 0, \\ \lambda(0) = \lambda_0, \quad \psi(0) = 0. \end{cases}$$

Moreover, $F^{-1}(0)$ near $(0,\lambda_0)$ consists precisely of the curves $x = 0$ and $(su_0 + s\psi(s), \lambda(s))$, $|s| < \delta$.

One generalization of the Crandall–Rabinowitz theorem is the following theorem which was proved by L. Nirenberg in [20] using Morse lemma.

Theorem B *(Nirenberg). Assume $F \in C^p(U \times \mathbb{R}, Y)$, $p \geqslant 2$, $F(0,\lambda_0) = 0$, $F_\lambda(0,\lambda_0) = 0$, $F_x(0,\lambda_0)$ is a Fredholm operator with* $\dim\ker F_x(0,\lambda_0) = \operatorname{codim}\operatorname{Im} F_x(0,\lambda_0) = 1$, $F_{\lambda\lambda}(0,\lambda_0) \in \operatorname{Im} F_x(0,\lambda_0)$, *and $F_{x\lambda}(0,\lambda_0)u_0 \notin \operatorname{Im} F_x(0,\lambda_0)$, where u_0 is such that* $\operatorname{span}\{u_0\} = \ker F_x(0,\lambda_0)$. *Then $(0,\lambda_0)$ is a bifurcation point of F, and the set of solutions of $F(x,\lambda) = 0$ near $(0,\lambda_0)$ consists of two C^{p-2} curves Γ_1, Γ_2 intersecting only at $(0,\lambda_0)$. Furthermore, if $p > 2$, then Γ_1 is tangent to the λ-axis at $(0,\lambda_0)$ and so may be parameterized by λ:*

$$\big(x(\lambda), \lambda\big), \quad |\lambda| \leqslant \varepsilon,$$

and Γ_2 may be parameterized by a variable s, $|s| \leqslant \varepsilon$, as

$$\big(su_0 + s\psi(s), \lambda(s)\big),$$

with $\psi(0) = 0$ and $\lambda(0) = \lambda_0$.

The Crandall–Rabinowitz theorem was also generalized in other directions; see, for instance, [18].

In case F satisfies $F(0,\lambda) = 0$ the curve Γ_1 is the λ-axis, that is $x = 0$, and therefore Theorem B generalizes Theorem A. For the elliptic boundary value problem

$$-\Delta u = \lambda u + g(x,u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $g(x,u) \in C^1$ with $g(x,0) = g'_u(x,0) = 0$, if λ_0 is a simple eigenvalue of $-\Delta$ with the Dirichlet boundary condition, then the Crandall–Rabinowitz theorem tells us that (θ,λ_0) is a bifurcation point and there exists a unique C^1 solution curve in $(C^{2,\gamma}(\Omega) \cap C_0(\bar{\Omega})) \times \mathbb{R}$, $0 < \gamma < 1$, emanating from $(0,\lambda_0)$.

In this paper, we will weaken the smoothness assumption on the operator F in the Crandall–Rabinowitz theorem and therefore give a new type of generalization. The main motivation for doing that lies in the study of bifurcation phenomenon of elliptic problems in which the Fučik spectrum plays an important role. In fact, our abstract theorem can be applied to problems related to the Fučik spectrum which involve non-smooth operators.

An elliptic boundary value problem much more complicated than (1.1) is

$$-\Delta u = au^- + bu^+ + g(x,u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{1.2}$$

where $u^- = \min\{0,u\}$, $u^+ = \max\{0,u\}$, $(a,b) \in \mathbb{R}^2$ is a pair of parameters, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function. Recall that $(a,b) \in \mathbb{R}^2$ is in the Fučik spectrum Σ of $-\Delta$ if and only if the boundary value problem

$$-\Delta u = au^- + bu^+ \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0 \tag{1.3}$$

has a nontrivial solution. For convenience, we also call the triple (u,a,b) a solution of (1.2) (resp. (1.3)) if (u,a,b) satisfies (1.2) (resp. (1.3)), and say (u,a,b) is nontrivial if $u \neq 0$.

The Fučik spectrum was first introduced by Fučik [14,15] and Dancer [7,8] in studying elliptic boundary value problems with jumping nonlinearities. Problems concerning the Fučik spectrum have since gained vast considerations. Note that the Fučik spectrum contains the two straight lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ and the points (λ_l, λ_l), where $\{\lambda_l\}_{l=1}^{\infty}$ are the eigenvalues of $-\Delta$ with 0-Dirichlet boundary condition.

In the situation of ordinary differential equations ($N = 1$), the Fučik spectrum has received extensive studies and it has a very clear picture in many cases; see, for instance, [7,8,13–15,24]. In particular, it is known that the Fučik spectrum Σ of the operator $-u''$ is the union of the two straight lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ as well as a sequence of decreasing curves, with one or two curves passing through (λ_l, λ_l) for each l, which have explicit expressions; see [8,15] or Section 5 below.

In the case of partial differential equations, it is proved in [12] that Σ contains a decreasing curve C asymptotic to the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ such that it passes through (λ_2, λ_2) and the intersection of Σ with the lower left side region of $\mathbb{R}^2 \setminus C$ is the union of the two lines. One does not know if there exists a global curve passing through (λ_l, λ_l) if $l \geqslant 3$ and, generally speaking, only local behaviors of Σ around (λ_l, λ_l), $l \geqslant 3$, have been obtained. It is shown in [26,27] that there exist two decreasing curves C_{l1}, C_{l2} (which may coincide) in $Q_l = (\lambda_{l-1}, \lambda_{l+1})^2$ passing through (λ_l, λ_l) for $l \geqslant 2$ such that all points on the curves are in Σ and there are no points in Σ inside Q_l above both curves or below both curves. Recently, it is proved in [17] that if λ_l with $l \geqslant 2$ is a simple eigenvalue and the corresponding eigenfunctions φ_l satisfy $\int_\Omega (\varphi_l^+)^2 \neq \int_\Omega (\varphi_l^-)^2$ then $\Sigma \cap Q_l = C_{l1} \cup C_{l2}$. Even though the Fučik spectrum of the partial differential operator $-\Delta$ has drawn much attention [1,4,9–12,16,17,22,21,25–28], its whole picture is far from clear yet. This makes (1.2) a difficult problem to solve.

We will study existence of two-dimensional bifurcations of solutions (u,a,b) of (1.2) related to the Fučik spectrum. We will assume that

$$g \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R}), \quad g(x,0) \equiv 0, \qquad g_t'(x,0) \equiv 0.$$

Denote $X = Z = C_0^1(\bar{\Omega})$ with the standard norm $\|\cdot\|$ and let $(a_0,b_0) \in \Sigma \setminus ((\mathbb{R} \times \{\lambda_1\}) \cup (\{\lambda_1\} \times \mathbb{R}))$. Then the boundary value problem

$$-\Delta u = a_0 u^- + b_0 u^+ \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

has a solution $u_0 \in X$ with $\|u_0\| = 1$ such that $u_0^+ \neq 0$ and $u_0^- \neq 0$. We will also assume that

$$\ker\bigl[\mathrm{id} - (-\Delta)^{-1}\bigl(a_0 A_{u_0}^- + b_0 A_{u_0}^+\bigr)\bigr] = \mathrm{span}\{u_0\}, \tag{1.4}$$

where the operators $A_u^\pm : L^p(\Omega) \to L^q(\Omega)$ with $p > q > N$ are defined as

$$A_u^\pm v(x) = \begin{cases} v(x) & \text{if } \pm u(x) > 0, \\ 0 & \text{if } \pm u(x) \leqslant 0. \end{cases}$$

The assumption (1.4) is a non-degenerate assumption which was first introduced in [22] and has been used in [17,25].

Denote $X_1 = \{v \in C_0^1(\bar{\Omega}):\ \int_\Omega \nabla u_0 \cdot \nabla v = 0\}$. To illustrate the kind of results we obtain, we state one result here.

Theorem 1.1. *Under the above assumptions, there exist $r, \tau > 0$ and a unique C^1 surface $(\psi, b) : (0, r) \times (a_0 - r, a_0 + r) \to B_{X_1}(\tau) \times (b_0 - \tau, b_0 + \tau)$ such that*

$$\lim_{s \to 0_+,\, a \to a_0} \psi(s, a) = 0, \qquad \lim_{s \to 0_+,\, a \to a_0} b(s, a) = b_0,$$

and $(su_0 + s\psi(s, a), a, b(s, a))$ is a solution of (1.2) for any $0 < s < r$ and $|a - a_0| < r$. Moreover the intersection of the set $\mathscr{S}$ of solutions of (1.2) with

$$\mathcal{C}(r, \tau) = \bigl\{(su_0 + sv, a, b):\ 0 \leqslant s < r,\ v \in B_{X_1}(\tau),\ |a - a_0| < r,\ |b - b_0| < \tau\bigr\}$$

is the union of $\mathcal{D}(r, \tau) = \{(0, a, b):\ |a - a_0| < r,\ |b - b_0| < \tau\}$ and $\mathcal{S}(r) = \{(su_0 + s\psi(s, a), a, b(s, a)):\ 0 < s < r,\ |a - a_0| < r\}$.

Theorem 1.1 reveals a two-dimensional bifurcation phenomenon for (1.2). This theorem cannot be a direct consequence of the classical abstract bifurcation theorems (for instance, the Crandall–Rabinowitz bifurcation theorem) since the classical theorems describe one-dimensional bifurcation phenomenon and require smoothness of the operators. We have to develop new techniques for proving Theorem 1.1. The main difficulty for doing this lies in the fact that the operators $u \mapsto u^\pm$ are not differentiable for every u. We will overcome this difficulty by proving a new variant implicit function theorem and a new variant abstract bifurcation theorem which only require the operators to be smooth for u in some dense subset of the whole space.

If $a = b$, then $\mathcal{S}(r)$ can be written as $\mathcal{S}(r) = \{(su_0 + s\psi(s), \lambda(s)):\ 0 < s < r\}$ and the result is the same as obtained by the Crandall–Rabinowitz bifurcation theorem.

The paper is organized as follows. We prove the new variant implicit function theorem in Section 2 and the new variant abstract bifurcation theorem in Section 3. In Section 4 the result in Section 3 is applied to (1.2) and several results including Theorem 1.1 are proved. Stronger results are obtained in Section 5 for ordinary differential equations.

2. An implicit function theorem

In this section, we will prove a variant implicit function theorem, which may be regarded as a generalization of the classical implicit function theorem. This new implicit function theorem will be used in Section 3 to deduce a bifurcation theorem, which will be applied to elliptic boundary value problems related to the Fučik spectrum. An implicit function theorem of the same kind was proved in [17].

Let X and Z be Banach spaces, and $u_0 \in X \setminus \{0\}$. Let X_1 be a closed subspace of X such that $X = X_1 \oplus \operatorname{span}\{u_0\}$. Let $0 < \rho \leqslant +\infty$ and $(a_0, b_0) \in \mathbb{R}^2$. Denote

$$D(\rho) = \{(a,b) \in \mathbb{R}^2\colon\ (a-a_0)^2 + (b-b_0)^2 < \rho^2\},$$
$$B_{X_1}(\rho) = \{v \in X_1\colon\ \|v\| < \rho\},$$

and

$$\mathcal{N} = \{u\colon\ u = v + su_0,\ v \in B_{X_1}(\rho),\ 0 \leqslant s < \rho\}.$$

Note that $\mathcal{N}$ is a cylinder with the point 0 being the center of its bottom $B_{X_1}(\rho)$. Let $F : \mathcal{N} \times D(\rho) \to Z$ be a mapping satisfying

$$F(0,a,b) = 0, \quad (a,b) \in D(\rho).$$

Consider the solution set of the equation

$$F(x,a,b) = 0$$

near $(0, a_0, b_0)$ in $\mathcal{N} \times D(\rho)$. The main result of this section is the following implicit function theorem, which will be used in the next section to prove a bifurcation theorem.

Theorem 2.1. *Assume*

- (F_1) *$F : \mathcal{N} \times D(\rho) \to Z$ is Lipschitz continuous*;
- (F_2) *there exists a subset $\mathcal{G}$ of $\mathcal{N}$ such that $\{u_1 \in B_{X_1}(\rho)\colon\ su_0 + su_1 \in \mathcal{G}\}$ is dense in $B_{X_1}(\rho)$ for any $0 < s < \rho$, $\mu u_1 + (1-\mu)u_2 \in \mathcal{G}$ for a.e. $\mu \in [0,1]$ if at least one of u_1 and u_2 is in $\mathcal{G}$, and the partial Fréchet derivative $F'_u(u,a,b)$ exists for any $u \in \mathcal{G}$ and $(a,b) \in D(\rho)$*;
- (F_3) *the limit $M_+ := \lim_{v \in X_1,\, su_0 + sv \in \mathcal{G},\, s \to 0_+,\, v \to 0,\, a \to a_0,\, b \to b_0} F'_u(su_0 + sv, a, b)|_{X_1}$, which is taken in the bounded linear operator space $\mathcal{L}(X_1, Z)$, exists and is an isomorphism from X_1 to Z*;
- (F_4) $\lim_{s \to 0_+,\, a \to a_0,\, b \to b_0} s^{-1} F(su_0, a, b) = 0$.

Then there exist $r, \tau \in (0, \rho)$ and a unique mapping $v : (0,r) \times D(r) \to B_{X_1}(\tau)$ such that

- (i) *$F(su_0 + sv(s,a,b), a, b) = 0$ for all $(s,a,b) \in (0,r) \times D(r)$*;
- (ii) *$v : (0,r) \times D(r) \to B_{X_1}(\tau)$ is Lipschitz continuous, and*

$$\lim_{s \to 0_+,\, a \to a_0,\, b \to b_0} v(s,a,b) = 0;$$

(iii) *if $F'_{(u,a,b)}$ exists and is continuous on $\mathcal{G} \times D(\rho)$, and $su_0 + sv(s,a,b) \in \mathcal{G}$ for any $(s,a,b) \in (0,r) \times D(r)$, then $v : (0,r) \times D(r) \to B_{X_1}(\tau)$ is of class C^1 and*

$$\begin{aligned} v'_s(s,a,b) &= -\frac{1}{s}\bigl[F'_u\bigl(su_0 + sv(s,a,b),a,b\bigr)\big|_{X_1}\bigr]^{-1} F'_u\bigl(su_0 + sv(s,a,b),a,b\bigr)u_0 \\ &\quad - \frac{1}{s}v(s,a,b), \\ v'_a(s,a,b) &= -\frac{1}{s}\bigl[F'_u\bigl(su_0 + sv(s,a,b),a,b\bigr)\big|_{X_1}\bigr]^{-1} F'_a\bigl(su_0 + sv(s,a,b),a,b\bigr), \\ v'_b(s,a,b) &= -\frac{1}{s}\bigl[F'_u\bigl(su_0 + sv(s,a,b),a,b\bigr)\big|_{X_1}\bigr]^{-1} F'_b\bigl(su_0 + sv(s,a,b),a,b\bigr). \end{aligned}$$

Proof. For $v \in B_{X_1}(\rho)$, $0 < s < \rho$, and $(a,b) \in D(\rho)$, define

$$g(v,s,a,b) = v - \frac{1}{s}M_+^{-1}F(su_0 + sv,a,b). \tag{2.1}$$

Then $g(v,s,a,b) \in X_1$. Set $N = \|M_+^{-1}\|_{\mathcal{L}(Z,X_1)}$. The conditions (F_2) and (F_3) imply the existence of $\tau > 0$ such that for $0 < s < \tau$, $v \in \overline{B_{X_1}(\tau)}$, and $(a,b) \in D(\tau)$, if $su_0 + sv \in \mathcal{G}$ then

$$\bigl\|F'_u(su_0 + sv,a,b)\big|_{X_1} - M_+\bigr\|_{\mathcal{L}(X_1,Z)} \leqslant \frac{1}{6N}. \tag{2.2}$$

Use (F_4) to find $r \in (0,\tau)$ such that for $0 < s < r$ and $(a,b) \in D(r)$,

$$\bigl\|F(su_0,a,b)\bigr\| < \frac{\tau s}{6N}. \tag{2.3}$$

Now fix $0 < s < r$ and $(a,b) \in D(r)$ and consider g as an operator from $B_{X_1}(\rho)$ to X_1. We apply the Banach fixed point theorem to g. First of all, (2.1) and (2.2) imply that if $v \in \overline{B_{X_1}(\tau)}$ and $su_0 + sv \in \mathcal{G}$ then

$$\bigl\|g'_v(v,s,a,b)\bigr\|_{\mathcal{L}(X_1,X_1)} = \bigl\|I - M_+^{-1}F'_u(su_0 + sv,a,b)\big|_{X_1}\bigr\|_{\mathcal{L}(X_1,X_1)} \leqslant \frac{1}{6}. \tag{2.4}$$

Let $v_1, v_2 \in \overline{B_{X_1}(\tau)}$ and $v_1 \neq v_2$. Using (F_1) and (F_2), we choose $\tilde{v}_1, \tilde{v}_2 \in \overline{B_{X_1}(\tau)}$, with

$$su_0 + s\tilde{v}_1, su_0 + s\tilde{v}_2 \in \mathcal{G},$$

such that

$$\|\tilde{v}_i - v_i\| < \|v_1 - v_2\|, \quad i = 1,2, \tag{2.5}$$

and

$$\bigl\|g(\tilde{v}_i,s,a,b) - g(v_i,s,a,b)\bigr\| < \frac{1}{6}\|v_1 - v_2\|, \quad i = 1,2. \tag{2.6}$$

Choose $\phi \in X_1^*$, $\|\phi\| = 1$, such that

$$\phi\big(g(\tilde{v}_1, s, a, b) - g(\tilde{v}_2, s, a, b)\big) = \big\|g(\tilde{v}_1, s, a, b) - g(\tilde{v}_2, s, a, b)\big\|. \tag{2.7}$$

Define

$$h(t) = \phi \circ g\big(\tilde{v}_2 + t(\tilde{v}_1 - \tilde{v}_2), s, a, b\big), \quad 0 \leqslant t \leqslant 1. \tag{2.8}$$

Then (F_1) implies that h is Lipschitz continuous. Since both $su_0 + s\tilde{v}_1$ and $su_0 + s\tilde{v}_2$ are in $\mathcal{G}$, (F_2) implies that, for a.e. $t \in [0, 1]$, $s(u_0 + \tilde{v}_2 + t(\tilde{v}_1 - \tilde{v}_2)) \in \mathcal{G}$ and therefore $g_v(\tilde{v}_2 + t(\tilde{v}_1 - \tilde{v}_2), s, a, b)$ exists and

$$h'(t) = \phi\big(g_v'\big(\tilde{v}_2 + t(\tilde{v}_1 - \tilde{v}_2), s, a, b\big)(\tilde{v}_1 - \tilde{v}_2)\big). \tag{2.9}$$

Using (2.4) and (2.7)–(2.9) in the formula $h(1) - h(0) = \int_0^1 h'(t)\,dt$, we deduce that

$$\big\|g(\tilde{v}_1, s, a, b) - g(\tilde{v}_2, s, a, b)\big\| \leqslant \frac{1}{6}\|\tilde{v}_1 - \tilde{v}_2\|. \tag{2.10}$$

Combining (2.5), (2.6), and (2.10) implies, for any $v_1, v_2 \in \overline{B_{X_1}(\tau)}$,

$$\big\|g(v_1, s, a, b) - g(v_2, s, a, b)\big\| \leqslant \frac{5}{6}\|v_1 - v_2\|. \tag{2.11}$$

From (2.1), (2.3), and (2.11), we see that, for any $v \in \overline{B_{X_1}(\tau)}$,

$$\big\|g(v, s, a, b)\big\| \leqslant \frac{5}{6}\|v\| + \big\|g(0, s, a, b)\big\| < \frac{5}{6}\tau + \frac{1}{6}\tau = \tau. \tag{2.12}$$

The Banach fixed point theorem together with (2.11) and (2.12) implies the existence of a unique solution $v = v(s, a, b) \in \overline{B_{X_1}(\tau)}$ to the equation $v = g(v, s, a, b)$. This means that there exists a unique mapping $v : (0, r) \times D(r) \to B_{X_1}(\tau)$ such that (i) holds.

To prove the Lipschitz continuity of v, assume $(s_1, a_1, b_1), (s_2, a_2, b_2) \in (0, r) \times D(r)$ and denote $v_i = v(s_i, a_i, b_i)$, $i = 1, 2$. Using (2.11) we see that

$$\begin{aligned}\|v_2 - v_1\| &\leqslant \big\|g(v_2, s_2, a_2, b_2) - g(v_1, s_2, a_2, b_2)\big\| + \big\|g(v_1, s_2, a_2, b_2) - g(v_1, s_1, a_1, b_1)\big\| \\ &\leqslant \frac{5}{6}\|v_2 - v_1\| + \big\|g(v_1, s_2, a_2, b_2) - g(v_1, s_1, a_1, b_1)\big\|,\end{aligned}$$

which implies

$$\begin{aligned}\|v_2 - v_1\| &\leqslant 6\big\|g(v_1, s_2, a_2, b_2) - g(v_1, s_1, a_1, b_1)\big\| \\ &\leqslant 6N\big\|s_2^{-1}F(s_2u_0 + s_2v_1, a_2, b_2) - s_1^{-1}F(s_1u_0 + s_1v_1, a_1, b_1)\big\|. \end{aligned} \tag{2.13}$$

Therefore, since F is Lipschitz continuous,

$$\lim_{(s_2,a_2,b_2)\to(s_1,a_1,b_1)} v(s_2,a_2,b_2)=v(s_1,a_1,b_1)$$

and $v:(0,r)\times D(r)\to B_{X_1}(\tau)$ is continuous. This property together with the Lipschitz continuity of F then implies the Lipschitz continuity of v. Use (2.1) and (2.12) to deduce that

$$\left\|v(s,a,b)\right\|\leqslant 6\left\|g(0,s,a,b)\right\|\leqslant 6N\frac{1}{s}\left\|F(su_0,a,b)\right\|,$$

which together with (F_4) implies

$$\lim_{s\to 0_+,\,a\to a_0,\,b\to b_0} v(s,a,b)=0. \tag{2.14}$$

Thus (ii) holds.

Now we turn to the first formula of (iii). Decreasing r if necessary we may assume that $F'_u(su_0+sv(s,a,b),a,b)|_{X_1}:X_1\to Z$ is an isomorphism for $0<s<r$ and $(a,b)\in D(r)$. Fix $s_1\in(0,r)$. For $s_2\in(0,r)$ with $s_2\neq s_1$ and $(a,b)\in D(r)$, denote $\tilde v_i=v(s_i,a,b)$, $i=1,2$. For simplicity of notations we denote, for $0\leqslant\theta\leqslant 1$,

$$\tilde w_\theta=s_1u_0+s_1\tilde v_1+\theta(s_2u_0+s_2\tilde v_2-s_1u_0-s_1\tilde v_1).$$

Since $su_0+sv(s,a,b)\in\mathcal{G}$ for any $(s,a,b)\in(0,r)\times D(r)$ and since $F(\tilde w_i,a,b)=0$ for $i=0,1$, we obtain

$$\int_0^1 F'_u(\tilde w_\theta,a,b)(\tilde w_1-\tilde w_0)\,d\theta=0. \tag{2.15}$$

Denote $N_1=\|[F'_u(\tilde w_0,a,b)|_{X_1}]^{-1}\|_{\mathcal{L}(Z,X_1)}$. Then

$$\begin{aligned}&\left\|\frac{\tilde v_2-\tilde v_1}{s_2-s_1}+\frac{1}{s_1}\left[F'_u(\tilde w_0,a,b)\big|_{X_1}\right]^{-1}F'_u(\tilde w_0,a,b)u_0+\frac{\tilde v_1}{s_1}\right\|\\&\quad\leqslant N_1\left\|F'_u(\tilde w_0,a,b)\left[\frac{\tilde v_2-\tilde v_1}{s_2-s_1}+\frac{u_0+\tilde v_1}{s_1}\right]\right\|.\end{aligned} \tag{2.16}$$

Divide (2.15) by $s_1(s_2-s_1)$ and then insert it into (2.16). We have

$$\begin{aligned}&\left\|\frac{\tilde v_2-\tilde v_1}{s_2-s_1}+\frac{1}{s_1}\left[F'_u(\tilde w_0,a,b)\big|_{X_1}\right]^{-1}F'_u(\tilde w_0,a,b)u_0+\frac{\tilde v_1}{s_1}\right\|\\&\quad\leqslant N_1\left\|\int_0^1 F'_u(\tilde w_\theta,a,b)\frac{\tilde w_1-\tilde w_0}{s_1(s_2-s_1)}\,d\theta-F'_u(\tilde w_0,a,b)\left[\frac{\tilde v_2-\tilde v_1}{s_2-s_1}+\frac{u_0+\tilde v_1}{s_1}\right]\right\|.\end{aligned}$$

Since

$$\frac{\tilde w_1-\tilde w_0}{s_1(s_2-s_1)}=\frac{\tilde v_2-\tilde v_1}{s_2-s_1}+\frac{u_0+\tilde v_2}{s_1},$$

it can be deduced that

$$\begin{aligned}
&\left\| \frac{\tilde{v}_2 - \tilde{v}_1}{s_2 - s_1} + \frac{1}{s_1}\left[F_u'(\tilde{w}_0, a, b)\big|_{X_1}\right]^{-1} F_u'(\tilde{w}_0, a, b)u_0 + \frac{\tilde{v}_1}{s_1} \right\| \\
&\quad \leqslant N_1 \int_0^1 \left\| F_u'(\tilde{w}_\theta, a, b) - F_u'(\tilde{w}_0, a, b) \right\|_{\mathcal{L}(X_1, Z)} d\theta \\
&\qquad \times \left\| \frac{\tilde{v}_2 - \tilde{v}_1}{s_2 - s_1} + \frac{1}{s_1}\left[F_u'(\tilde{w}_0, a, b)\big|_{X_1}\right]^{-1} F_u'(\tilde{w}_0, a, b)u_0 + \frac{\tilde{v}_1}{s_1} \right\| \\
&\qquad + N_1 \left\| \int_0^1 F_u'(\tilde{w}_\theta, a, b)\, d\theta \left[\frac{\tilde{v}_2 - \tilde{v}_1}{s_1} + \frac{u_0}{s_1} - \frac{1}{s_1}\left[F_u'(\tilde{w}_0, a, b)\big|_{X_1}\right]^{-1} F_u'(\tilde{w}_0, a, b)u_0 \right] \right\|.
\end{aligned}$$

Using continuity of v and F_u', we can find $\delta > 0$ such that if $0 < |s_2 - s_1| < \delta$ then

$$\int_0^1 \left\| F_u'(\tilde{w}_\theta, a, b) - F_u'(\tilde{w}_0, a, b) \right\|_{\mathcal{L}(X_1, Z)} d\theta \leqslant \frac{1}{2N_1},$$

and thus

$$\begin{aligned}
&\left\| \frac{\tilde{v}_2 - \tilde{v}_1}{s_2 - s_1} + \frac{1}{s_1}\left[F_u'(\tilde{w}_0, a, b)\big|_{X_1}\right]^{-1} F_u'(\tilde{w}_0, a, b)u_0 + \frac{\tilde{v}_1}{s_1} \right\| \\
&\quad \leqslant 2N_1 \left\| \int_0^1 F_u'(\tilde{w}_\theta, a, b)\, d\theta \left[\frac{\tilde{v}_2 - \tilde{v}_1}{s_1} + \frac{u_0}{s_1} - \frac{1}{s_1}\left[F_u'(\tilde{w}_0, a, b)\big|_{X_1}\right]^{-1} F_u'(\tilde{w}_0, a, b)u_0 \right] \right\|.
\end{aligned}$$

Using continuity of F_u' and v again, we obtain

$$\lim_{s_2 \to s_1} \left\{ \frac{\tilde{v}_2 - \tilde{v}_1}{s_2 - s_1} + \frac{1}{s_1}\left[F_u'(\tilde{w}_0, a, b)\big|_{X_1}\right]^{-1} F_u'(\tilde{w}_0, a, b)u_0 + \frac{\tilde{v}_1}{s_1} \right\} = 0.$$

This proves the first formula of (iii). The proofs of the other two formulas of (iii) are similar but easier, and are dropped. The C^1 regularity of v then follows from these formulas. □

The main new feature of Theorem 2.1 is that F need not be Fréchet differentiable with respect to u on the whole $\mathcal{N} \times D(\rho)$. It is only required that for some suitable subset $\mathcal{G}$ of $\mathcal{N}$, F is Fréchet differentiable with respect to u on $\mathcal{G} \times D(\rho)$. Theorem 2.1 is in particular useful when one study equations involving nonlinear operators which are not smooth everywhere.

The assumption (F_4) in Theorem 2.1 implies that there exists $\delta > 0$ such that if $|a - a_0| + |b - b_0| < \delta$ then $F(0, a, b) = 0$. Therefore it seems to be redundant to state the assumption $F(0, a, b) = 0$ explicitly before the theorem, and we did in that way for emphasis.

Since $\mathcal{G}$ is just a subset of $\mathcal{N}$, the assumption that $su_0 + sv(s, a, b) \in \mathcal{G}$ for any $(s, a, b) \in (0, r) \times D(r)$ in Theorem 2.1(iii) seems to be too strong and one may doubt that it be satisfied in applications. Of course, this assumption is hard to verify in dealing with abstract problems.

But we will see that with our concrete elliptic boundary value problems (1.2) and (1.3) it is quite natural and is indeed satisfied.

The following lemma is also a variant implicit function theorem. It will also be used in the next section in proving the bifurcation theorem. Its proof is standard and thus is omitted.

Lemma 2.2. *Assume* $h \in C^1((0,\rho) \times D(\rho), \mathbb{R})$, $\lim_{s\to 0_+,\, a\to a_0,\, b\to b_0} h(s,a,b) = 0$, *and* $\lim_{s\to 0_+,\, a\to a_0,\, b\to b_0} h'_b(s,a,b)$ *exists and is a nonzero real number. Then there exist* $r, \tau \in (0,\rho)$ *and a unique* C^1 *function* $b : (0,r) \times (a_0 - r, a_0 + r) \to (b_0 - \tau, b_0 + \tau)$ *such that*

$$\lim_{s\to 0_+,\, a\to a_0} b(s,a) = b_0$$

and

$$h\big(s,a,b(s,a)\big) = 0 \quad \textit{for all } (s,a) \in (0,r) \times (a_0 - r, a_0 + r).$$

3. A bifurcation theorem

In this section, we will use the implicit function theorem from Section 2 to prove a two-dimensional bifurcation theorem. Let X, Z, X_1, u_0, (a_0,b_0), $D(\rho)$, $B_{X_1}(\rho)$, and $\mathcal{N}$ be as in Section 2. Let $f : \mathcal{N} \times D(\rho) \to Z$ be a mapping such that

$$f(0,a,b) = 0 \quad \text{for all } (a,b) \in D(\rho).$$

We will use the following assumptions.

- (f_1) $f : \mathcal{N} \times D(\rho) \to Z$ is Lipschitz continuous;
- (f_2) there exists a subset $\mathcal{G}$ of $\mathcal{N}$ such that $\{u_1 \in B_{X_1}(\rho)\colon\ su_0 + su_1 \in \mathcal{G}\}$ is dense in $B_{X_1}(\rho)$ for $0 < s < \rho$, $\mu u_1 + (1-\mu)u_2 \in \mathcal{G}$ for a.e. $\mu \in [0,1]$ if at least one of u_1 and u_2 is in $\mathcal{G}$, $su_0 \in \mathcal{G}$ for $0 < s < \rho$, the partial Fréchet derivative $f'_u(u,a,b)$ exists and is continuous on $\mathcal{G} \times D(\rho)$, and the partial Fréchet derivative $f'_{(a,b)}(u,a,b)$ exists and is continuous on $\mathcal{N} \times D(\rho)$;
- (f_3) the limit $\tilde{M}_+ := \lim_{v\in X_1,\, su_0+sv\in\mathcal{G},\, s\to 0_+,\, v\to 0,\, a\to a_0,\, b\to b_0} f'_u(su_0+sv,a,b)$, which is taken in the bounded linear operator space $\mathcal{L}(X,Z)$, exists and is a Fredholm operator of index 0, and $\ker \tilde{M}_+ = \operatorname{span}\{u_0\}$;
- (f_4) the limit

$$w_+ := \lim_{v\in X_1,\, su_0+sv\in\mathcal{G},\, s\to 0_+,\, v\to 0,\, a\to a_0,\, b\to b_0} \frac{1}{s} f'_b(su_0+sv,a,b)$$

  exists and $w_+ \notin \operatorname{range} \tilde{M}_+$.

The assumption (f_3) implies the existence of $z_0 \in Z$, $z_0 \neq 0$, such that Z can be decomposed as

$$Z = \operatorname{span}\{z_0\} \oplus \operatorname{range} \tilde{M}_+.$$

Let $\mathcal{P} : Z \to \operatorname{range} \tilde{M}_+$ be the projection with respect to this decomposition. Combining (f_2) and (f_3) yields

$$\frac{1}{s} f(su_0, a, b) = \int_0^1 f'_u(\theta s u_0, a, b) u_0 \, d\theta \to \tilde{M}_+ u_0 = 0, \tag{3.1}$$

as $s \to 0_+$, $a \to a_0$, and $b \to b_0$.

Applying Theorem 2.1 to the operator $F = \mathcal{P} f : \mathcal{N} \times D(\rho) \to \operatorname{range} \tilde{M}_+$, we have the following lemma as a consequence.

Lemma 3.1. *Assume (f_1)–(f_3) hold. Then there exist $r, \tau \in (0, \rho)$ and a unique mapping $v : (0, r) \times D(r) \to B_{X_1}(\tau)$ such that*

(i) $\mathcal{P} f(su_0 + sv(s, a, b), a, b) = 0$ *for all* $(s, a, b) \in (0, r) \times D(r)$;
(ii) $v : (0, r) \times D(r) \to B_{X_1}(\tau)$ *is Lipschitz continuous, and*

$$\lim_{s \to 0_+,\, a \to a_0,\, b \to b_0} v(s, a, b) = 0;$$

(iii) *if* $su_0 + sv(s, a, b) \in \mathcal{G}$ *for any* $(s, a, b) \in (0, r) \times D(r)$, *then* $v : (0, r) \times D(r) \to B_{X_1}(\tau)$ *is of class* C^1 *and*

$$\begin{aligned} v'_s(s, a, b) &= -\frac{1}{s} \big[\mathcal{P} f'_u\big(su_0 + sv(s, a, b), a, b\big)\big|_{X_1} \big]^{-1} \mathcal{P} f'_u\big(su_0 + sv(s, a, b), a, b\big) u_0 \\ &\quad - \frac{1}{s} v(s, a, b), \\ v'_a(s, a, b) &= -\frac{1}{s} \big[\mathcal{P} f'_u\big(su_0 + sv(s, a, b), a, b\big)\big|_{X_1} \big]^{-1} \mathcal{P} f'_a\big(su_0 + sv(s, a, b), a, b\big), \\ v'_b(s, a, b) &= -\frac{1}{s} \big[\mathcal{P} f'_u\big(su_0 + sv(s, a, b), a, b\big)\big|_{X_1} \big]^{-1} \mathcal{P} f'_b\big(su_0 + sv(s, a, b), a, b\big), \end{aligned}$$

where $[\mathcal{P} f'_u(su_0 + sv(s, a, b), a, b)|_{X_1}]^{-1}$ *is the inverse of the operator* $\mathcal{P} f'_u(su_0 + sv(s, a, b), a, b)|_{X_1} : X_1 \to \operatorname{range} \tilde{M}_+$.

Note that in the setting of Lemma 3.1, for $(s, a, b) \in (0, r) \times D(r)$, the mapping g in the proof of Theorem 2.1, which yields $v(s, a, b)$ as the unique solution of the equation $v = g(v, s, a, b)$ in $B_{X_1}(\tau)$, is defined by

$$g(v, s, a, b) = v - \frac{1}{s} (\mathcal{P} \tilde{M}_+|_{X_1})^{-1} \mathcal{P} f(su_0 + sv, a, b). \tag{3.2}$$

The main result of this section is the following theorem, which will be applied in the next section to the study of two-dimensional bifurcation phenomenon of elliptic boundary value problems. To state the result, we make the following assumption.

(f_5) $su_0 + sv(s, a, b) \in \mathcal{G}$ for any $(s, a, b) \in (0, r) \times D(r)$, where v is given by Lemma 3.1.

As noted in Section 2, this assumption is natural regarding its verification in applications to elliptic boundary value problems.

Theorem 3.2. *Suppose f satisfies (f_1)–(f_5). Then $(0, a_0, b_0) \in X \times \mathbb{R}^2$ is a bifurcation point, and there exist $r, \tau > 0$ and a unique C^1 surface $(\psi, b) : (0, r) \times (a_0 - r, a_0 + r) \to B_{X_1}(\tau) \times (b_0 - \tau, b_0 + \tau)$ such that*

$$\lim_{s \to 0_+,\, a \to a_0} \psi(s, a) = 0, \qquad \lim_{s \to 0_+,\, a \to a_0} b(s, a) = b_0, \tag{3.3}$$

$$f\big(su_0 + s\psi(s, a), a, b(s, a)\big) = 0 \quad \textit{for } (s, a) \in (0, r) \times (a_0 - r, a_0 + r), \tag{3.4}$$

and

$$\begin{aligned} f^{-1}(0) \cap \mathcal{C} = {} & \big\{(0, a, b)\colon\ |a - a_0| < r,\ |b - b_0| < \tau\big\} \\ & \cup \big\{\big(su_0 + s\psi(s, a), a, b(s, a)\big)\colon\ 0 < s < r,\ |a - a_0| < r\big\}, \end{aligned} \tag{3.5}$$

where

$$\mathcal{C} = \big\{(su_0 + sv, a, b)\colon\ 0 \leqslant s < r,\ v \in B_{X_1}(\tau),\ |a - a_0| < r,\ |b - b_0| < \tau\big\}.$$

Proof. For any $z \in Z$, write $z = \alpha z_0 + z_1$ with $\alpha \in \mathbb{R}$ and $z_1 \in \operatorname{range} \tilde{M}_+$. Define $\phi \in Z^*$ by $\phi(z) = \alpha$. Then the equation $f(u, a, b) = 0$ is equivalent to the system of the two equations

$$\mathcal{P} f(u, a, b) = 0, \qquad \big\langle \phi, f(u, a, b)\big\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between Z^* and Z. According to Lemma 3.1, there exist $r_1, \tau_1 \in (0, \rho)$ and $v : (0, r_1) \times D(r_1) \to B_{X_1}(\tau_1)$ such that the conclusions of Lemma 3.1 hold for r_1 and τ_1 in place of r and τ respectively. Then for $(s, a, b) \in (0, r_1) \times D(r_1)$, $(su_0 + sv(s, a, b), a, b)$ satisfies the first equation of the above system, and therefore the equation

$$f\big(su_0 + sv(s, a, b), a, b\big) = 0$$

is reduced to

$$h(s, a, b) := \frac{1}{s}\big\langle \phi, f\big(su_0 + sv(s, a, b), a, b\big)\big\rangle = 0. \tag{3.6}$$

For fixed (s, a), we will solve (3.6) for $b = b(s, a)$. Using (f_2), (f_5), and Lemma 3.1(iii), we see that $h \in C^1((0, r_1) \times D(r_1), \mathbb{R})$. Moreover, (f_1) gives a constant $L > 0$ such that

$$\frac{1}{s}\big\| f\big(su_0 + sv(s, a, b), a, b\big)\big\| \leqslant L\big\| v(s, a, b)\big\| + \frac{1}{s}\big\| f(su_0, a, b)\big\|. \tag{3.7}$$

Now (3.6) and (3.7) combined with (3.1) and Lemma 3.1(ii) imply

$$\lim_{s \to 0_+,\, a \to a_0,\, b \to b_0} h(s, a, b) = 0. \tag{3.8}$$

Note from (f_2), (f_5), and Lemma 3.1(iii) that

$$h'_b(s,a,b) = \langle \phi, f'_u\big(su_0 + sv(s,a,b),a,b\big)\big(v'_b(s,a,b)\big)\rangle + \left\langle \phi, \frac{1}{s} f'_b\big(su_0 + sv(s,a,b),a,b\big)\right\rangle.$$

Moreover, (f_3), (f_5), and Lemma 3.1(ii) imply

$$\lim_{s\to 0_+,\, a\to a_0,\, b\to b_0} f'_u\big(su_0 + sv(s,a,b),a,b\big) = \tilde{M}_+,$$

while the definition of w_+ and Lemma 3.1(ii) yield

$$\lim_{s\to 0_+,\, a\to a_0,\, b\to b_0} \frac{1}{s} f'_b\big(su_0 + sv(s,a,b),a,b\big) = w_+.$$

Now use (f_3), Lemma 3.1(ii), and the last formula of Lemma 3.1(iii) to deduce

$$\lim_{s\to 0_+,\, a\to a_0,\, b\to b_0} v'_b(s,a,b) = -[\mathcal{P}\tilde{M}_+|_{X_1}]^{-1}\mathcal{P}w_+.$$

Therefore, since $\ker\phi = \operatorname{range}\tilde{M}_+$, (f_4) implies that

$$\lim_{s\to 0_+,\, a\to a_0,\, b\to b_0} h'_b(s,a,b) = \langle \phi, -\mathcal{P}w_+\rangle + \langle \phi, w_+\rangle = \langle \phi, w_+\rangle \neq 0. \tag{3.9}$$

As a consequence of (3.8) and (3.9), the assumptions of Lemma 2.2 are satisfied by h. Therefore, for some $r \in (0,r_1)$ and $\tau \in (0,\tau_1)$ with $r^2 + \tau^2 < r_1^2$, there exists a unique C^1 function $b : (0,r) \times (a_0 - r, a_0 + r) \to (b_0 - \tau, b_0 + \tau)$ such that

$$\lim_{s\to 0_+,\, a\to a_0} b(s,a) = b_0$$

and

$$h\big(s,a,b(s,a)\big) = 0 \quad \text{for all } (s,a) \in (0,r) \times (a_0 - r, a_0 + r).$$

Define $\psi(s,a) = v(s,a,b(s,a))$. Since both v and b are C^1, so is the mapping $\psi : (0,r) \times (a_0 - r, a_0 + r) \to B_{X_1}(\tau_1)$. Decreasing r if necessary, we may assume that ψ maps $(0,r) \times (a_0 - r, a_0 + r)$ into $B_{X_1}(\tau)$. Obviously, the second equation in (3.3) and (3.4) hold. Using Lemma 3.1(ii) we then obtain the first equation in (3.3). Therefore, there exists a unique C^1 surface $(\psi, b) : (0,r) \times (a_0 - r, a_0 + r) \to B_{X_1}(\tau) \times (b_0 - \tau, b_0 + \tau)$ such that (3.3) and (3.4) are satisfied. Now (3.5) follows from the uniqueness of (ψ, b). □

Remark 3.3. If $\lambda = a = b$ and $f \in C^2$, then Theorem 3.2 reduces to the Crandall–Rabinowitz theorem.

4. Bifurcation from the Fučik spectrum

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the elliptic boundary value problem

$$\begin{cases} -\Delta u = au^- + bu^+ + g(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

and the positively homogeneous boundary value problem

$$-\Delta u = au^- + bu^+ \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{4.2}$$

In this section, we will study existence of two-dimensional bifurcations related to the Fučik spectrum. Denote $X = Z = C_0^1(\bar{\Omega})$ with the standard norm denoted by $\|\cdot\|$. Fix two numbers p, q such that $p > q > N$. For $u \in X$, define two operators $A_u^\nu : L^p(\Omega) \to L^q(\Omega)$ with $\nu = +$ or $-$ by

$$A_u^\pm v(x) = \begin{cases} v(x) & \text{if } \pm u(x) > 0, \\ 0 & \text{if } \pm u(x) \leqslant 0. \end{cases}$$

These operators were introduced in [17] in studying the Fučik spectrum of $-\Delta$.

Let $a_0, b_0 > \lambda_1$ be two numbers. We formulate the assumptions for (4.1).

(g_1) $g \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, $g(x,0) \equiv 0$, $g_t'(x,0) \equiv 0$;
(g_2) $u_0 \in X$, $\|u_0\| = 1$, and $\ker[\mathrm{id} - (-\Delta)^{-1}(a_0 A_{u_0}^- + b_0 A_{u_0}^+)] = \mathrm{span}\{u_0\}$.

Note that (g_2) implies that u_0 is a nontrivial solution of the boundary value problem

$$-\Delta u = a_0 u^- + b_0 u^+ \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{4.3}$$

Thus (a_0, b_0) is in the Fučik spectrum Σ. Moreover, since $a_0, b_0 > \lambda_1$, u_0 is a sign-changing solution, that is, $u_0^+ \neq 0$ and $u_0^- \neq 0$. (g_2) is a local non-degenerate assumption at (u_0, b_0, a_0) which was first introduced in [22] in studying the Fučik spectrum. The global version of (g_2) was later used in [25], where it was showed that this assumption is a generic assumption. According to [17, Theorem 3.5], if λ_l is simple, $a_0, b_0 \in (\lambda_{l-1}, \lambda_{l+1})$, and $(a_0, b_0) \in \Sigma$ then the local non-degenerate assumption holds.

Denote $X_1 = \{v \in C_0^1(\bar{\Omega})\colon \int_\Omega \nabla u_0 \cdot \nabla v = 0\}$. Then $X = X_1 \oplus \mathrm{span}\{u_0\}$.

Theorem 4.1. *Suppose (g_1)–(g_2) are satisfied. Then there exist $r, \tau > 0$ and a unique C^1 surface $(\psi, b) : (0, r) \times (a_0 - r, a_0 + r) \to B_{X_1}(\tau) \times (b_0 - \tau, b_0 + \tau)$ such that*

$$\lim_{s \to 0_+, a \to a_0} \psi(s,a) = 0, \qquad \lim_{s \to 0_+, a \to a_0} b(s,a) = b_0,$$

and $(su_0 + s\psi(s,a), a, b(s,a))$ is a solution of (4.1) for any $0 < s < r$ and $|a - a_0| < r$. Moreover the intersection of the set $\mathscr{S}$ of solutions of (4.1) with

$$\mathcal{C}(r,\tau) = \big\{(su_0 + sv, a, b)\colon 0 \leqslant s < r,\ v \in B_{X_1}(\tau),\ |a - a_0| < r,\ |b - b_0| < \tau\big\}$$

is the union of $\mathcal{D}(r,\tau)=\{(0,a,b)\colon |a-a_0|<r,\ |b-b_0|<\tau\}$ *and* $\mathcal{S}(r)=\{(su_0+s\psi(s,a), a, b(s,a))\colon 0<s<r,\ |a-a_0|<r\}$.

Proof. Define $f: X\times\mathbb{R}^2\to Z$ by

$$f(u,a,b)=u-(-\Delta)^{-1}\big(au^-+bu^++g(x,u)\big). \tag{4.4}$$

Then (u,a,b) is a solution of (4.1) if and only if $f(u,a,b)=0$. Obviously, (g_1) implies that $f: X\times\mathbb{R}^2\to Z$ is Lipschitz continuous and thus (f_1) is satisfied. Define

$$\mathcal{G}=\{u\in X\colon\ u\neq 0 \text{ a.e. in } \Omega\}.$$

By [17, Lemmas 3.1 and 3.2], (f_2) holds. According to [17], if $su_0+sv\in\mathcal{G}$ then

$$\begin{aligned} f_u'(su_0+sv,a,b)&=\mathrm{id}-(-\Delta)^{-1}\big(aA^+_{su_0+sv}+bA^-_{su_0+sv}+g_t'(x,su_0+sv)\big)\\ &=\mathrm{id}-(-\Delta)^{-1}\big(aA^+_{u_0+v}+bA^-_{u_0+v}+g_t'(x,su_0+sv)\big). \end{aligned}$$

Now, if $su_0+sv\in\mathcal{G}$ then using (g_1) and the proof of [17, Lemmas 3.2] we see that

$$\lim_{v\in X_1,\, su_0+sv\in\mathcal{G},\, s\to 0_+,\, v\to 0,\, a\to a_0,\, b\to b_0} f_u'(su_0+sv,a,b)=\mathrm{id}-(-\Delta)^{-1}\big(a_0A^+_{u_0}+b_0A^-_{u_0}\big),$$

where the limit is taken in $\mathcal{L}(X,X)$. Denote $\tilde{M}_+=\mathrm{id}-(-\Delta)^{-1}(a_0A^+_{u_0}+b_0A^-_{u_0})$. Then $\tilde{M}_+$ is a Fredholm operator of index 0 and (g_2) implies (f_3). Note that

$$w_+:=\lim_{v\in X_1,\, su_0+sv\in\mathcal{G},\, s\to 0_+,\, v\to 0,\, a\to a_0,\, b\to b_0}\frac{1}{s}f_b'(su_0+sv,a,b)=-(-\Delta)^{-1}u_0^+.$$

We claim $w_+\notin\operatorname{range}\tilde{M}_+$. Assume to the contrary $w_+\in\operatorname{range}\tilde{M}_+$. Then there exists $u\in X$ such that

$$-\Delta u=a_0A^+_{u_0}u+b_0A^-_{u_0}u-u_0^+\quad\text{in }\Omega,\qquad u|_{\partial\Omega}=0.$$

Multiplying the last equation with u_0 and taking integral by parts yields

$$\int_\Omega\big(u_0^+\big)^2=0,$$

which contradicts the fact that $u_0^+\neq 0$. Therefore, (f_4) is valid. In order to apply Theorem 3.2, it remains to check (f_5). Set $z_0=(-\Delta)^{-1}u_0$. Then we deduce that $z_0\notin\operatorname{range}\tilde{M}_+$, just as above. Define the projection $\mathcal{P}: Z\to\operatorname{range}\tilde{M}_+$ according to the space decomposition $Z=\operatorname{span}\{z_0\}\oplus\operatorname{range}\tilde{M}_+$. Let $v:(0,r)\times D(r)\to B_{X_1}(\tau)$ be given as in Lemma 3.1. Set $u=su_0+sv(s,a,b)$ for $(s,a,b)\in(0,r)\times D(r)$ for simplicity of notation. Then since $\mathcal{P}f(su_0+sv(s,a,b),a,b)=0$, there exists $\alpha\in\mathbb{R}$ such that

$$-\Delta u=au^-+bu^++g(x,u)+\alpha u_0\quad\text{in }\Omega,\qquad u|_{\partial\Omega}=0. \tag{4.5}$$

If $\alpha = 0$ then there exists $c > 0$ such that $|\Delta u| \leqslant c|u|$ and unique continuation of solutions of elliptic equations (see, for instance, [2] or [19, Lemma 3.1]) implies that $u \neq 0$ a.e. in Ω, and thus $u \in \mathcal{G}$. If $\alpha \neq 0$ we also claim $u \in \mathcal{G}$. Suppose to the contrary that $u = 0$ on a subset of Ω with positive measure, that is, $|\{x \in \Omega\colon u(x) = 0\}| > 0$. The implicit function theorem implies that

$$\left|\left\{x \in \Omega\colon u(x) = 0,\ \nabla u(x) \neq 0\right\}\right| = 0,$$

and

$$\left|\left\{x \in \Omega\colon \nabla u(x) = 0,\ \nabla^2 u(x) \neq 0\right\}\right| = 0.$$

Therefore,

$$\begin{aligned}\left|\left\{x \in \Omega\colon u(x) = 0,\ \nabla u(x) = 0,\ \nabla^2 u(x) = 0\right\}\right| &= \left|\left\{x \in \Omega\colon u(x) = 0,\ \nabla u(x) = 0\right\}\right| \\ &= \left|\left\{x \in \Omega\colon u(x) = 0\right\}\right| > 0,\end{aligned}$$

which implies $|\{x \in \Omega\colon u(x) = 0,\ \Delta u(x) = 0\}| > 0$. This property together with (4.5) implies $|\{x \in \Omega\colon u_0(x) = 0\}| > 0$. But u_0 is a nonzero solution of (4.3). The unique continuation property says that $|\{x \in \Omega\colon u_0(x) = 0\}| = 0$, and we obtain a contradiction. Therefore in any case $u \in \mathcal{G}$ and (f_5) is satisfied. Now the result follows from Theorem 3.2. □

Note that the parameters a and b play equal roles in (4.1). Exchanging a and b in Theorem 4.1, we also have the following result.

Theorem 4.2. *Suppose (g_1)–(g_2) are satisfied. Then there exist $r, \tau > 0$ and a unique C^1 surface $(\psi, a) : (0, r) \times (b_0 - r, b_0 + r) \to B_{X_1}(\tau) \times (a_0 - \tau, a_0 + \tau)$ such that*

$$\lim_{s \to 0_+,\, b \to b_0} \psi(s, b) = 0, \qquad \lim_{s \to 0_+,\, b \to b_0} a(s, b) = a_0,$$

and $(su_0 + s\psi(s, b), a(s, b), b)$ is a solution of (4.1) for any $0 < s < r$ and $|b - b_0| < r$. Moreover the intersection of the set $\mathscr{S}$ of solutions of (4.1) with

$$\hat{\mathcal{C}}(r, \tau) = \left\{(su_0 + sv, a, b)\colon 0 \leqslant s < r,\ v \in B_{X_1}(\tau),\ |a - a_0| < \tau,\ |b - b_0| < r\right\}$$

is the union of $\hat{\mathcal{D}}(r, \tau) = \{(0, a, b)\colon |a - a_0| < \tau,\ |b - b_0| < r\}$ and $\hat{\mathcal{S}}(r) = \{(su_0 + s\psi(s, b), a(s, b), b)\colon 0 < s < r,\ |b - b_0| < r\}$.

Now we consider the special case where $g(x, t) \equiv 0$. From (4.4), for $s > 0$, $v \in X_1$, and $(a, b) \in \mathbb{R}^2$, we have

$$f(su_0 + sv, a, b) = s\left[u_0 + v - (-\Delta)^{-1}\left(a(u_0 + v)^- + b(u_0 + v)^+\right)\right],$$

and therefore $\frac{1}{s} f(su_0 + sv, a, b)$ is independent of s. The operator g in (3.2) takes the form

$$g(v, s, a, b) = v - (\mathcal{P}\tilde{M}_+|X_1)^{-1}\mathcal{P} f(u_0 + v, a, b),$$

and $g(v,s,a,b)$ is independent of s. Thus $v(s,a,b)$, as a solution of the equation $v = g(v,s,a,b)$, obtained by Lemma 3.1 via Theorem 2.1 is independent of s. Checking the proof of Theorem 3.2 we see that $h(s,a,b)$ defined in (3.6) is independent of s. For $0 < s < r$ and $a \in (a_0 - r, a_0 + r)$, $b = b(s,a)$ as the unique solution of the equation $h(s,a,b) = 0$ in $(b_0 - \tau, b_0 + \tau)$ is independent of s. Therefore $\psi(s,a) = v(s,a,b(s,a))$ is independent of s. Denote $\psi(a) = \psi(s,a)$ and $b(a) = b(s,a)$. We then have the following corollary of Theorem 4.1.

Corollary 4.3. *Suppose (g_2) is satisfied. Then there exist $r, \tau > 0$ and a unique C^1 surface $(\psi, b): (a_0 - r, a_0 + r) \to B_{X_1}(\tau) \times (b_0 - \tau, b_0 + \tau)$ such that*

$$\lim_{a \to a_0} \psi(a) = 0, \qquad \lim_{a \to a_0} b(a) = b_0,$$

and $(su_0 + s\psi(a), a, b(a))$ is a solution of (4.2) for any $0 < s < r$ and $|a - a_0| < r$. Moreover the intersection of the set $\mathscr{S}$ of solutions of (4.2) with

$$\mathcal{C}(r,\tau) = \big\{(su_0 + sv, a, b)\colon\ 0 \leqslant s < r,\ v \in B_{X_1}(\tau),\ |a - a_0| < r,\ |b - b_0| < \tau\big\}$$

is the union of $\mathcal{D}(r,\tau) = \{(0,a,b)\colon\ |a - a_0| < r,\ |b - b_0| < \tau\}$ and $\mathcal{S}(r) = \{(su_0 + s\psi(a), a, b(a))\colon\ 0 < s < r,\ |a - a_0| < r\}$.

If we define $u(a) = (u_0 + \psi(a))/\|u_0 + \psi(a)\|_{H_0^1(\Omega)}$, then we recover the first part of [17, Theorem 3.3] from Corollary 4.3.

The projection of $\mathcal{C}(r,\tau)$ into X is the set $\mathscr{C} = \{su_0 + sv\colon\ 0 \leqslant s < r,\ v \in B_{X_1}(\tau)\}$, which is a cone in X with vertex 0, axis $\{su_0\colon\ 0 \leqslant s < r\}$, and base $\{u_0 + v\colon\ v \in B_{X_1}(\tau)\}$. Theorems 4.1, 4.2, and Corollary 4.3 describe the solution set of (4.1) and (4.2) in $\mathcal{C}(r,\tau)$ whose projections into X is the cone $\mathscr{C}$. In special cases, we can also describe the solution set of (4.1) and (4.2) in sets whose projections into X are neighborhoods of 0, as the following two corollaries show.

Corollary 4.4. *Suppose (g_1)–(g_2) are satisfied. If $u_0 \in X$, with $\|u_0\| = 1$, is the unique solution of (4.3), then there exist $r^* \in (0,r)$ and $\tau^* \in (0,\tau)$ such that the intersection of the set $\mathscr{S}$ of solutions of (4.1) with*

$$\mathcal{Q}\big(r^*,\tau^*\big) = \big\{(su_0 + v, a, b)\colon\ |s| < r^*,\ v \in B_{X_1}\big(r^*\tau^*\big),\ |a - a_0| < r^*,\ |b - b_0| < \tau^*\big\}$$

is the union of $\mathcal{D}(r^*,\tau^*)$ and $\mathcal{S}(r^*)$.

Proof. Obviously, $\mathscr{S} \cap \mathcal{Q}(r^*,\tau^*) \supset \mathcal{D}(r^*,\tau^*) \cup \mathcal{S}(r^*)$ for any $r^* \in (0,r)$ and $\tau^* \in (0,\tau)$. We claim the former is also contained in the latter if r^* and τ^* are small enough. Assume to the contrary that there exists a sequence of solutions $\{(s_n u_0 + v_n, a_n, b_n)\}$ of (4.1) such that $(s_n u_0 + v_n, a_n, b_n) \in \mathcal{Q}(1/n, 1/n) \setminus (\mathcal{D}(1/n,1/n) \cup \mathcal{S}(1/n))$. Set $t_n = \|s_n u_0 + v_n\|$, which is positive since $s_n u_0 + v_n \neq 0$. Define

$$w_n = \frac{1}{t_n}(s_n u_0 + v_n).$$

Then

$$w_n = (-\Delta)^{-1}\left(a_n w_n^- + b_n w_n^+ + \frac{1}{t_n} g(x, s_n u_0 + v_n)\right).$$

Using (g_1) we see that $\{w_n\}$ is compact in X. Passing to a subsequence we may assume that $w_n \to w$ in X, $\|w\| = 1$, and w is a solution of (4.3). Then $w = u_0$ by the uniqueness assumption on the solution of (4.3), and $w_n \to u_0$. Decompose w_n as $w_n = \alpha_n u_0 + (w_n - \alpha_n u_0)$ so that $\alpha_n \in \mathbb{R}$ and $w_n - \alpha_n u_0 \in X_1$. Note that $t_n \to 0$ and $\alpha_n \to 1$. In view of

$$s_n u_0 + v_n = t_n w_n = t_n \alpha_n \left(u_0 + \left(\frac{1}{\alpha_n} w_n - u_0\right)\right),$$

we see that, for n large, $s_n = t_n \alpha_n > 0$ and $(s_n u_0 + v_n, a_n, b_n) \in \mathcal{C}(r, \tau)$, where r and τ are given in Theorem 4.1. Since $s_n > 0$, we obtain

$$v_n = s_n \psi(s_n, a_n), \qquad b_n = b(s_n, a_n).$$

Since $(s_n u_0 + v_n, a_n, b_n) \in \mathcal{Q}(1/n, 1/n)$ we have $0 < s_n < \frac{1}{n}$ and $|a_n - a_0| < \frac{1}{n}$. The definition of $\mathcal{S}(r)$ yields $(s_n u_0 + v_n, a_n, b_n) \in \mathcal{S}(\frac{1}{n})$, which is a contradiction. □

To state the next corollary, we need to assume that

(g_3) the boundary value problem (4.3) has exactly k solutions $u_i \in X$ with $\|u_i\| = 1$, and moreover $\ker[\mathrm{id} - (-\Delta)^{-1}(a_0 A_{u_i}^- + b_0 A_{u_i}^+)] = \mathrm{span}\{u_i\}$, $i = 1, 2, \ldots, k$.

Denote $X_i = \{v \in C_0^1(\bar{\Omega})\colon \int_\Omega \nabla u_i \cdot \nabla v = 0\}$ for $i = 1, 2, \ldots, k$.

Corollary 4.5. *Suppose (g_1) and (g_3) are satisfied. Then there exist $r, \tau > 0$ such that for each $i \in \{1, \ldots, k\}$ there exists exactly one C^1 surface $(\psi_i, b_i) : (0, r) \times (a_0 - r, a_0 + r) \to B_{X_i}(\tau) \times (b_0 - \tau, b_0 + \tau)$ such that*

$$\lim_{s \to 0_+, a \to a_0} \psi_i(s, a) = 0, \qquad \lim_{s \to 0_+, a \to a_0} b_i(s, a) = b_0,$$

and $(su_i + s\psi_i(s, a), a, b_i(s, a))$ is a solution of (4.1) for any $0 < s < r$ and $|a - a_0| < r$. Moreover there exist $r^* \in (0, r)$, $\tau^* \in (0, \tau)$ and a neighborhood U of 0 in X such that the intersection of the set $\mathscr{S}$ of solutions of (4.1) with

$$\mathcal{B}(r^*, \tau^*) = \{(u, a, b)\colon u \in U,\ |a - a_0| < r^*,\ |b - b_0| < \tau^*\}$$

is $\mathcal{D}(r^*, \tau^*) \cup (\bigcup_{1 \leqslant i \leqslant k} \mathcal{S}_i(r^*))$ where

$$\mathcal{S}_i(r^*) = \{(su_i + s\psi_i(s, a), a, b_i(s, a))\colon 0 < s < r^*,\ |a - a_0| < r^*,\ su_i + s\psi_i(s, a) \in U\}.$$

Proof. The first part of the conclusion clearly holds. For the second part of the conclusion, we obviously have $\mathscr{S} \cap \mathcal{B}(r^*, \tau^*) \supset \mathcal{D}(r^*, \tau^*) \cup (\bigcup_{1 \leqslant i \leqslant k} \mathcal{S}_i(r^*))$ for any $r^* \in (0, r)$ and $\tau^* \in (0, \tau)$. If by contradiction the two sets are not equal for any $r^* \in (0, r)$ and $\tau^* \in (0, \tau)$, then there exists a sequence of solutions $\{(\hat{u}_n, a_n, b_n)\}$ of (4.1) such that $0 \neq \hat{u}_n \to 0$, $a_n \to a_0$, and $b_n \to b_0$. Denote $w_n = \hat{u}_n / \|\hat{u}_n\|$. Then w_n satisfies

$$w_n = (-\Delta)^{-1}\big(a_n w_n^- + b_n w_n^+ + g(x, \hat{u}_n)/\|\hat{u}_n\|\big).$$

As in the proof of Corollary 4.4, passing to a subsequence we may assume that $w_n \to w$ in X, $\|w\| = 1$, and w is a solution of (4.3). Then $w = u_i$ and $w_n \to u_i$ for some $i \in \{1, 2, \ldots, k\}$, according to the assumption (g_3). Chose $\alpha_n \in \mathbb{R}$ such that $w_n - \alpha_n u_i \in X_i$. Then $\alpha_n \to 1$. Since $0 < s_n := \|\hat{u}_n\| \alpha_n \to 0$ and

$$\hat{u}_n = \|\hat{u}_n\| w_n = \|\hat{u}_n\| \alpha_n \left(u_i + \left(\frac{1}{\alpha_n} w_n - u_i\right)\right),$$

we must have $\frac{1}{\alpha_n} w_n - u_i = \psi_i(s_n, a_n)$ and $b_n = b_i(s_n, a_n)$ according to the uniqueness of (ψ_i, b_i). Now we arrive at

$$(\hat{u}_n, a_n, b_n) = \big(s_n\big(u_i + \psi_i(s_n, a_n)\big), a_n, b_i(s_n, a_n)\big) \in \mathcal{S}_i\big(r^*\big)$$

for n large, which is a contradiction. □

5. The ODE case

As a special case, we consider in this section the two-point boundary value problem

$$-u'' = au^- + bu^+ + g(x, u) \quad \text{in } (0, \pi), \quad u(0) = u(\pi) = 0. \tag{5.1}$$

The eigenvalues of $-u'' = \lambda u$ subject to the boundary condition $u(0) = u(\pi) = 0$ are $\{n^2\}_{n=1}^{\infty}$. The entire picture of the Fučik spectrum Σ in this case is easy to describe, and it is known that Σ consists of the following curves, the first two being straight lines (see, for example, [14]),

$$c_l^1\colon \ \frac{l-1}{\sqrt{a}} + \frac{l}{\sqrt{b}} = 1, \qquad c_l^2\colon \ \frac{l}{\sqrt{a}} + \frac{l-1}{\sqrt{b}} = 1, \qquad c_l^3\colon \ \frac{l}{\sqrt{a}} + \frac{l}{\sqrt{b}} = 1, \quad l = 1, 2, \ldots.$$

Denote $P_n = (n^2, n^2)$. Note that c_l^1 and c_l^2 pass through P_{2l-1} and c_l^3 passes through P_{2l}. We claim that if $(a, b) \in (c_l^1 \cup c_l^2) \setminus \{P_{2l-1}\}$ then there exists exactly one $u_0 \in X$ such that (g_2) is satisfied. To see this we assume $(a, b) \in c_l^1 \setminus \{P_{2l-1}\}$ for definiteness. Denote for $i = 1, 2, \ldots,$

$$t_i = \frac{(i-1)\pi}{\sqrt{a}} + \frac{(i-1)\pi}{\sqrt{b}}, \qquad s_i = \frac{(i-1)\pi}{\sqrt{a}} + \frac{i\pi}{\sqrt{b}}, \qquad r_i = \frac{i\pi}{\sqrt{a}} + \frac{(i-1)\pi}{\sqrt{b}}.$$

The two-point boundary value problem

$$-u'' = au^- + bu^+ \quad \text{in } (0, \pi), \quad u(0) = u(\pi) = 0 \tag{5.2}$$

has a unique solution u up to normalization, whose zeros in $[0,\pi]$ are

$$0 = t_1 < s_1 < t_2 < s_2 < \cdots < t_l < s_l = \pi$$

and which is expressed by

$$u(t) = \begin{cases} \sqrt{a}\sin(\sqrt{b}(t - t_i)), & t_i \leqslant t \leqslant s_i \text{ for } i = 1, 2, \ldots, l, \\ -\sqrt{b}\sin(\sqrt{a}(t - s_i)), & s_i \leqslant t \leqslant t_{i+1} \text{ for } i = 1, 2, \ldots, l-1. \end{cases}$$

Moreover, it is straightforward to see that the solution set of the two-point boundary value problem

$$-v'' = aA_u^- v + bA_u^+ v \quad \text{in } (0,\pi), \quad v(0) = v(\pi) = 0 \tag{5.3}$$

is a one-dimensional linear space spanned by u.

Now we prove that (g_3) holds for $(a,b) \in c_l^3 \cup \{P_{2l-1}\}$. If $(a,b) \in c_l^3$ then (5.2) has exactly two solutions up to normalization, which are given by

$$u_1(t) = \begin{cases} \sqrt{a}\sin(\sqrt{b}(t - t_i)), & t_i \leqslant t \leqslant s_i \text{ for } i = 1, 2, \ldots, l, \\ -\sqrt{b}\sin(\sqrt{a}(t - s_i)), & s_i \leqslant t \leqslant t_{i+1} \text{ for } i = 1, 2, \ldots, l \end{cases}$$

and

$$u_2(t) = \begin{cases} -\sqrt{a}\sin(\sqrt{b}(t - t_i)), & t_i \leqslant t \leqslant r_i \text{ for } i = 1, 2, \ldots, l, \\ \sqrt{b}\sin(\sqrt{a}(t - r_i)), & r_i \leqslant t \leqslant t_{i+1} \text{ for } i = 1, 2, \ldots, l. \end{cases}$$

Moreover, the solution set of (5.3) with u_i in place of u is a one-dimensional linear space spanned by u_i. Thus (g_3) holds. If $(a,b) = P_{2l-1}$ then (5.2) has exactly two solutions u and $-u$ up to normalization, and in the same way (g_3) holds.

Combining the discussions above and the results in Section 4, we have the following two theorems.

Theorem 5.1. *Suppose (g_1) is satisfied. If $(a,b) \in \bigcup_{l=2}^{\infty}((c_l^1 \cup c_l^2) \setminus \{P_{2l-1}\})$ then there exist $r, \tau > 0$ and a unique C^1 surface $(\psi, b) : (0,r) \times (a_0 - r, a_0 + r) \to B_{X_1}(\tau) \times (b_0 - \tau, b_0 + \tau)$ such that*

$$\lim_{s \to 0_+,\, a \to a_0} \psi(s,a) = 0, \qquad \lim_{s \to 0_+,\, a \to a_0} b(s,a) = b_0,$$

and $(su_0 + s\psi(s,a), a, b(s,a))$ is a solution of (5.1) for any $0 < s < r$ and $|a - a_0| < r$. Moreover there exist $r^* \in (0,r)$ and $\tau^* \in (0,\tau)$ such that the intersection of the set $\mathscr{S}$ of solutions of (5.1) with

$$\mathcal{Q}(r^*,\tau^*) = \{(su_0 + v, a, b):\ |s| < r^*,\ v \in B_{X_1}(r^*\tau^*),\ |a - a_0| < r^*,\ |b - b_0| < \tau^*\}$$

is the union of $\mathcal{D}(r^*,\tau^*)$ and $\mathcal{S}(r^*)$.

Theorem 5.2. *Suppose* (g_1) *is satisfied. If* $(a,b) \in (\bigcup_{l=1}^{\infty} c_l^3) \cup \{P_{2l-1}\}_{l=2}^{\infty}$ *then there exist* $r, \tau > 0$ *and* 2 C^1 *surfaces* $(\psi_i, b_i) : (0,r) \times (a_0 - r, a_0 + r) \to B_{X_i}(\tau) \times (b_0 - \tau, b_0 + \tau)$ *such that*

$$\lim_{s \to 0_+,\, a \to a_0} \psi_i(s,a) = 0, \qquad \lim_{s \to 0_+,\, a \to a_0} b_i(s,a) = b_0,$$

and $(su_i + s\psi_i(s,a), a, b_i(s,a))$ *is a solution of* (5.1) *for any* $0 < s < r$, $|a - a_0| < r$, *and* $i = 1, 2$. *Moreover there exist* $r^* \in (0, r)$, $\tau^* \in (0, \tau)$ *and a neighborhood* U *of* 0 *in* X *such that the intersection of the set* $\mathscr{S}$ *of solutions of* (5.1) *with*

$$\mathcal{B}(r^*, \tau^*) = \{(u, a, b)\colon\ u \in U,\ |a - a_0| < r^*,\ |b - b_0| < \tau^*\}$$

is $\mathcal{D}(r^*, \tau^*) \cup (\bigcup_{1 \leqslant i \leqslant 2} \mathcal{S}_i(r^*))$ *where*

$$\mathcal{S}_i(r^*) = \{(su_i + s\psi_i(s,a), a, b_i(s,a))\colon\ 0 < s < r^*,\ |a - a_0| < r^*,\ su_i + s\psi_i(s,a) \in U\}.$$

Theorems 5.1 and 5.2 draw clear pictures of the solution sets of (5.1) in a neighborhood of $(0, a, b)$ in $X \times \mathbb{R}^2$ for all $(a, b) \in \Sigma$.

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